Lecture Note on Linear Algebra

16. Eigenvalues and Eigenvectors

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1 What Do You Learn from This Note

In this lecture note, we are considering a very special matrix equation for a given square matrix A:

$$A\vec{x} = \lambda\vec{x}. \quad (1)$$

This equation is very important for developing optimisation algorithm for many engineering problems(大家将来遇到很多的科学计算优化问题都可以归结为解这个特征向量方程的问题).

Basic Concept： Eigenvalue(特征值), Eigenvector(特征向量), Characteristic equation(特征方程)

2 Eigenvalue & Eigenvector

Definition 1 (EIGENVALUE(特征值) AND EIGENVECTOR(特征向量))**.** Let A be a $n \times n$ square matrix. An eigenvalue λ of A is a scalar such that $A\vec{x} = \lambda\vec{x}$ for some non–zero vector $\vec{x}$, which is called an eigenvector of A corresponding to λ (or a λ–eigenvector).

Example: For any $\vec{x} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, we have $(\lambda I_n)\vec{x} = \lambda\vec{x}$. So λ is an eigenvalue of λI_n and any non–zero vector is a λ–eigenvector.

Example: Let $A \in \mathbb{R}^n$ be non–invertible, which is equivalent to $\mathrm{Nul}(A) - \{\vec{0}\} \neq \emptyset$. Then we have $A\vec{x} = \vec{0} = 0\vec{x}$ for any $\vec{x} \in \mathrm{Nul}(A) - \{\vec{0}\}$. So 0 is an eigenvalue of A and any non–zero vector in $\text{Nul}(A) - \{\vec{0}\}$ is a 0–eigenvector.

Warning: By definition, eigenvalue can take zero scalar whereas eigenvector is restricted to non–zero vectors.(特征值可以是0，但特征向量不能是零向量)

Examples: Textbook P.303.

Theorem 2. *If $\vec{v}_1, \cdots, \vec{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \cdots, \lambda_r$ of an $n \times n$ matrix A, the set $\vec{v}_1, \cdots, \vec{v}_r$ is linearly independent.*

Proof. We conduct the proof by contradiction (我们用反正法) and induction method.

1. STEP 1: We first prove for two eigenvectors $\vec{v}_{i_1}$ and $\vec{v}_{i_2}$ that correspond to different eigenvalues $\lambda_{i_1}, \lambda_{i_2}$, $\vec{v}_{i_1}$ and $\vec{v}_{i_2}$ must be linearly independent. If not, then we have $\vec{v}_{i_1} = c\vec{v}_{i_2}$ for some $c \neq 0$. Also, $A\vec{v}_{i_1} = cA\vec{v}_{i_2}$, hence $\lambda_{i_1}\vec{v}_{i_1} = c\lambda_{i_2}\vec{v}_{i_2}$. Combing with $\lambda_{i_1}\vec{v}_{i_1} = c\lambda_{i_1}\vec{v}_{i_2}$, so $(\lambda_{i_1} - \lambda_{i_2})\vec{v}_{i_2} = \vec{0}$. Since $\lambda_{i_1} \neq \lambda_{i_2}$, hence $\vec{v}_{i_2} = \vec{0}$, which is impossible, as an eigenvector must be non-zero.

2. STEP 2: for any r-1 eigenvector, $\vec{v}_{i_1}, \cdots, \vec{v}_{i_{r-1}}$ corresponding to $r - 1$ distinct eigenvalues are linearly independent.

3. SETP 3: for given r eigenvectors, If these r vectors are linearly dependent, then there exists a vector $\vec{v}_i$ such that $\vec{v}_i$ is a linear combination of the other eigenvectors, that is there are a series of weights $c_1, c_2, \cdots, c_{i-1}, c_{i+1}, \cdots, c_r$ some of which are not zero having

$$\vec{v}_i = c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1} + c_{i+1}\vec{v}_{i+1} + \cdots + c_r\vec{v}_r. \tag{2}$$

Also by multiplying A on both side, we have

$$\lambda_i\vec{v}_i = c_1\lambda_1\vec{v}_1 + \cdots + \lambda_{i-1}c_{i-1}\vec{v}_{i-1} + c_{i+1}\lambda_{i+1}\vec{v}_{i+1} + \cdots + c_r\lambda_r\vec{v}_r. \tag{3}$$

In addition, by multiplying λ_i on both sides of Eq. (2), then we have

$$\lambda_i\vec{v}_i = c_1\lambda_i\vec{v}_1 + \cdots + c_{i-1}\lambda_i\vec{v}_{i-1} + c_{i+1}\lambda_i\vec{v}_{i+1} + \cdots + c_r\lambda_i\vec{v}_r. \tag{4}$$

Combining Eqs. 3 and 5, then we have

$$\vec{0} = c_1(\lambda_i - \lambda_1)\vec{v}_1 + \cdots + c_{i-1}(\lambda_i - \lambda_{i-1})\vec{v}_{i-1} + c_{i+1}(\lambda_i - \lambda_{i+1})\vec{v}_{i+1} + \cdots + c_r(\lambda_i - \lambda_r)\vec{v}_r. \tag{5}$$

As the $r-1$ eigenvectors $\vec{v}_1, \cdots, \vec{v}_{i-1}, \vec{v}_{i+1}, \cdots, \vec{v}_r$ are linearly independent, and not all the weights $c_1, c_2, \cdots, c_{i-1}, c_{i+1}, \cdots, c_r$ are zero, then there must be some weight $c_k \neq 0$ so that $\lambda_i - \lambda_k$. This contradicts to the statement that all the eigenvectors are distinct. Hence all the r eigenvectors that correspond to r different eigenvalues are linearly independent.

□

3 Computation of Eigenvalues and Eigenvectors

Lemma 3. Let A be a $n \times n$ square matrix. Then for any scalar λ, we have

$$\{\vec{x} \mid A\vec{x} = \lambda\vec{x}\} = \mathrm{Nul}(\lambda I_n - A).$$

Proof.

$$\begin{aligned}\{\vec{x} \mid A\vec{x} = \lambda\vec{x}\} &= \{\vec{x} \mid \lambda\vec{x} - A\vec{x} = \vec{0}\} \\ &= \{\vec{x} \mid (\lambda I_n)\vec{x} - A\vec{x} = \vec{0}\} \\ &= \{\vec{x} \mid (\lambda I_n - A)\vec{x} = \vec{0}\} \\ &= \mathrm{Nul}(\lambda I_n - A).\end{aligned}$$

□

Theorem 4. Let λ be an eigenvalue of a matrix A. Then all the λ–eigenvectors of A together with the zero vector form a subspace, which is called the **eigenspace** of A corresponding to λ (or the λ–eigenspace).

Proof. Since the set of all λ–eigenvectors together with $\vec{0}$ is exactly $\{\vec{x} \mid A\vec{x} = \lambda\vec{x}\} = \mathrm{Nul}(\lambda I_n - A)$, which is of course a subspace. □

Example: Textbook P.304.

Question? How to compute the eigenvalues?

Theorem 5. Let A be an $n \times n$ square matrix. Then the following statements are equivalent:
1. A scalar λ is an eigenvalue of A.
2. $\mathrm{Nul}(\lambda I_n - A) \neq \{\vec{0}\}$.
3. $\dim \mathrm{Nul}(\lambda I_n - A) \geqslant 1$.
4. $\lambda I_n - A$ is not invertible.
5. $\det(\lambda I_n - A) = 0$.

Proof. Since λ is an eigenvalue of A iff $\mathrm{Nul}(\lambda I_n - A)$ contains a non–zero vector iff $\mathrm{Nul}(\lambda I_n - A) \neq \{\vec{0}\}$. So 1. $\Leftrightarrow$ 2., and 2. $\Leftrightarrow$ 3. $\Leftrightarrow$ 4. $\Leftrightarrow$ 5. is straightforward. $\square$

Example: Find all the eigenvalues of $A = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$ and the corresponding eigenspaces.

Solution. Suppose that λ is an eigenvalue of A. Then by Theorem 4, $\det(\lambda I_n - A) = 0$, that is $\det \begin{pmatrix} \lambda - 3 & 1 \\ -2 & \lambda \end{pmatrix} = 0$. By expanding this determinant, we get a quadratic equation $\lambda^2 - 3\lambda + 2 = 0$ in λ. By solving this equation, we obtain all the eigenvalues of A are 1 and 2.
The 1–eigenspace is $\mathrm{Nul}(I_2 - A)$, that is the solution set of the homogenous equation $(I_2 - A)\vec{x} = \vec{0}$. By solving this equation, we obtain the 1–eigenspace is $\mathrm{Span}\{(1\ 2)^T\}$.
Similarly, we can find the 2–eigenspace, which is $\mathrm{Span}\{(1\ 1)^T\}$.

By the Theorem 5, we can easily have:

Theorem 6. *The eigenvalues of a triangular matrix are the entries on its main diagonal.*

Motivated by the above example of finding eigenvalues, we have

Definition 7 (CHARACTERISTIC EQUATION(特征方程))**.** Let A be a $n \times n$ square matrix. The characteristic equation of A is defined to be $\det(\lambda I_n - A) = 0$.

Remarks: The eigenvalues of A are precisely the roots of $c_A(\lambda)$ in the set of scalars. Hence the number of distinct eigenvalues must be lest than n.

Example: Textbook P.313.

Example: Find all the eigenvalues of $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and the corresponding eigenspaces over $\mathbb{R}$ and $\mathbb{C}$ respectively.

Solution. 1. $A \in \mathbb{R}^{2\times 2}$.
The characteristic polynomial $c_A(\lambda) = \det \begin{pmatrix} \lambda & 1 \\ -1 & \lambda \end{pmatrix} = \lambda^2 + 1$. Since $c_A(\lambda)$ has no real root, so no eigenvalue exists in $\mathbb{R}$.
2. $A \in \mathbb{C}^{2\times 2}$.
In this case, c_A has roots i and $-$i, which are all eigenvalues of A.
The i–eigenspace is $\mathrm{Nul}(\mathrm{i}I_2 - A)$, which can be computed by solving equation $(\mathrm{i}I_2 - A)\vec{x} = 0$ and the result is $\mathrm{Span}\{(\mathrm{i}\ 1)^T\}$. Similarly, we can obtain the $(-\mathrm{i})$–eigenspace $\mathrm{Span}\{(-\mathrm{i}\ 1)^T\}$.
This example illustrates that the existence of eigenvalues depends on the set of scalars (which is in fact a field).

Summary of the procedure of computing eigenvalues.

For a given square matrix A, its eigenvalues and the corresponding eigenspaces are computed as follows:
1. Expand $c_A(\lambda) = \det(\lambda I_n - A)$;
2. Solve $c_A(\lambda) = 0$ in the set of scalars ($\mathbb{R}$ or $\mathbb{C}$ in this course) to obtain all the eigenvalues, say $\lambda_1, \ldots, \lambda_r$.
3. For each $i = 1, \ldots, r$, solve the homogenous equation $(\lambda_i I_n - A)\vec{x} = \vec{0}$ and the solution set is the λ_i–eigenspace.

4 Diagonalization

We are now considering a special matrix factorization for some square matrix using eigenvectors and eigenvalues, which has the following forms

$$A = PDP^{-1}. \tag{6}$$

where P is an invertible matrix and D is a diagonal matrix. This factorization enables us to compute A^k by

$$A^k = PD^kP^{-1}. \tag{7}$$

We are now formally introducing the diagonalization processing.

Definition 8 (Similarity)**.** *Two square matrix A and B are similar if there is an invertible matrix P such that $P^{-1}AP = B$, or say equivalently $A = PBP^{-1}$. Changing A into $P^{-1}AP$ is called a similarity transformation.*

Definition 9 (Diagonalizable Matrix)**.** Let A be a square matrix. A is said to be diagonalisable if A is similar to a diagonal matrix D or equivalently, there exists an invertible matrix P such that $P^{-1}AP$ is diagonal.

Theorem 10. *If two $n \times n$ matrices A and B are similar, they have the same characteristic polynomial and hence the same eigenvalues.*

Proof. Since A and B are similar, there exists an invertible matrix P such that $B = P^{-1}AP$. Therefore, we have

$$B - \lambda I = P^{-1}AP - \lambda P^{-1}IP = P^{-1}(A - \lambda I)P,$$

So

$$\begin{aligned} \det B &= \det(P^{-1}(A - \lambda I)P) \\ &= \det P^{-1} \det(A - \lambda I) \det P \\ &= \det P^{-1} \det P \det(A - \lambda I) \\ &= \det(P^{-1}P) \det(A - \lambda I) \\ &= \det I \det(A - \lambda I) \\ &= \det(A - \lambda I). \end{aligned} \tag{8}$$

□

Not all square matrices are diagonalisable, then we have the following question:

Question: When can a square matrix A be diagonalized?

Theorem 11. Let $A \in \mathbb{R}^{n\times n}$. A is diagonalisable iff there exists a basis $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ such that $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent eigenvectors of A.

Proof. We prove the theorem by the following steps:
STEP 1. Suppose that A is diagonalisable. Then there exists an invertible matrix $P = (\vec{v}_1 \ \cdots \ \vec{v}_n)$ and a diagonal matrix $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$ such that $AP = PD$. So

$$(A\vec{v}_1 \ \cdots \ A\vec{v}_n) = AP = PD = (\lambda_1\vec{v}_1 \ \cdots \ \lambda_n\vec{v}_n).$$

So for all i, $\vec{v}_i$ is a λ_i–eigenvector of A, also the invertibility of P ensures that $\{\vec{v}_1, \ldots, \vec{v}_n\}$ forms a basis.
STEP 2. Conversely, let $P = (\vec{v}_1 \ \cdots \ \vec{v}_n)$. For $i = 1, \ldots, n$, let λ_i be the eigenvalue with eigenvector $\vec{v}_i$. Thus we have

$$AP = P\text{diag}(\lambda_1, \ldots, \lambda_n).$$

Since the eigenvectors are linearly independent, P is invertible, thus we have:

$$A = P\text{diag}(\lambda_1, \ldots, \lambda_n)P^{-1}.$$

□

Remark: The basis $\vec{v}_1 \ \cdots \ \vec{v}_n$ in the above theorem is called an **eigenvector basis**.

Corollary 12. If $A \in \mathbb{R}^{n\times n}$ has n distinct eigenvalues then A is diagonalisable.

Proof. By Theorem 2, $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is linearly independent, where for $i = 1, \ldots, n$, $\vec{v}_i$ is some λ_i–eigenvector. So $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis and A is diagonalisable by by Theorem 9. □

Question: Can we still perform diagonalization when not all eigenvalues are distinct?

Theorem 13. Let $\lambda_1, \ldots, \lambda_r$ be distinct eigenvalues of $A \in \mathbb{R}^{n\times n}$. For $i = 1, \ldots, r$, let $S_i = \{\vec{v}_{i1}, \ldots, \vec{v}_{in_i}\}$ be a linearly independent set of λ_i–eigenspace. Then $S = \bigcup_{i=1}^r S_i$ is linearly independent.

Proof. For $i = 1, \ldots, r$, let $c_{i1}, \ldots, c_{in_i}$ be scalars and $\vec{w}_i = c_{i1}\vec{v}_{i1} + \cdots + c_{in_i}\vec{v}_{in_i}$. Suppose that $\vec{w}_1 + \cdots + \vec{w}_r = \vec{0}$. We need to show $c_{i1} = \cdots = c_{in_i} = 0$ for all $i = 1, \ldots, r$. Now for $i = 1, \ldots, r$, $\vec{w}_i$ is either $\vec{0}$ or a λ_i–eigenvector. But $\{\vec{w}_1, \ldots, \vec{w}_r\}$ is linearly dependent, which forces that $\vec{w}_1 = \cdots = \vec{w}_r = \vec{0}$, otherwise contradicting the result of THEOREM 2. So for all $i = 1, \ldots, r$, $c_{i1} = \cdots = c_{in_i} = 0$ since S_i is linearly independent. □

Corollary 14. Let $\lambda_1, \ldots, \lambda_r$ be all distinct eigenvalues of $A \in \mathbb{R}^{n\times n}$ and n_i the dimension of λ_i–eigenspace for $i = 1, \ldots, r$. Then A is diagonalisable iff $\sum_{i=1}^r n_i = n$.

Proof. Suppose A is similar to D, where D is diagonal. Then it is not hard to see that A and D have the same set of eigenvalues and the dimensions of the corresponding eigenspaces are equal (Exercise). But for D, hence for A, we have $\sum_{i=1}^r n_i = n$.
Conversely, suppose that $\mathcal{B}_i$ is a basis of λ_i–eigenspace for $i = 1, \ldots, r$. Then $\mathcal{B} = \bigcup_{i=1}^r \mathcal{B}_i$ is linearly independent by THEOREM 13. Also $|\mathcal{B}| = \sum_{i=1}^r n_i = n$. So $\mathcal{B}$ is a basis consisting of eigenvectors. So A is diagonalisable by THEOREM 11. □

According to COROLLARY 14, the following procedure can be used to determined whether or not a given $A \in \mathbb{M}_n$ is diagonalisable and in the affirmative case, a matrix P such that $P^{-1}AP$ is diagonal is computed.
STEP 1: Compute all eigenvalues $\lambda_1, \ldots, \lambda_r$ of A.
STEP 2: For each $i = 1, \ldots, r$, compute a basis $\mathcal{B}_i$ of the λ_i–eigenspace.
STEP 3: If $\sum_{i=1}^r |\mathcal{B}_i| \neq n$, then A is not diagonalisable. Otherwise, the matrix $P = (\vec{v}_1 \ \cdots \ \vec{v}_n)$, where $\{\vec{v}_1, \ldots, \vec{v}_n\} = \bigcup_{i=1}^r \mathcal{B}_i$, satisfies that $P^{-1}AP$ is diagonal.

Example: Textbook P.324.

4.1 A Linear Transformation View of Diagolization

Objective: We aim to understand the diagonalization of matrix A from the linear transformation view of point (在本节，我们尝试利用线性变换解释对矩阵A的对角话操作).

STEP 1: Linear Transformation between Vector Space V and W

Let T be the linear transformation between V and W. Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \cdots, \vec{b}_n\}$ be a basis of vector space V. Let $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \cdots, \vec{c}_m\}$ be a basis of vector space W.

Then any $\vec{x} \in V$ can be represented by a linear combination of $\vec{b}_1, \vec{b}_2, \cdots, \vec{b}_n$ with weight $r_1, r_2, \cdots, r_n$ as follows:

$$\vec{x} = r_1\vec{b}_1 + \cdots + r_n\vec{b}_n = B[\vec{x}]_{\mathcal{B}}.$$

where $B = [\vec{b}_1, \vec{b}_2, \cdots, \vec{b}_n]$ and $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}$. Then by the definition of linear transformation, we have

$$\begin{aligned} T(\vec{x}) &= T(r_1\vec{b}_1 + \cdots + r_n\vec{b}_n) \\ &= r_1 T(\vec{b}_1) + \cdots + r_n T(\vec{b}_n) \end{aligned} \tag{9}$$

So, the coordinate of $T(\vec{x})$ under basis C in W is

$$\begin{aligned} [T(\vec{x})]_{\mathcal{C}} &= r_1[T(\vec{b}_1)]_{\mathcal{C}} + \cdots + r_n[T(\vec{b}_n)]_{\mathcal{C}} \\ &= M[x]_{\mathcal{B}}, \end{aligned} \tag{10}$$

where $M = \big([T(\vec{b}_1)]_{\mathcal{C}}, \cdots, [T(\vec{b}_n)]_{\mathcal{C}}\big)$.

We call matrix M as the **matrix for T relative to the bases $\mathcal{B}$ to $\mathcal{C}$**.

STEP 2: Linear Transformation from V to V (i.e. $W = V$). In this case, the M matrix is called the **matrix for T relative to $\mathcal{B}$**, or the **$\mathcal{B}$-matrix for T**. We re-denote this $\mathcal{B}$-matrix by $[T]_{\mathcal{B}}$ as follows:

$$[T]_{\mathcal{B}} = \big([T(\vec{b}_1)]_{\mathcal{B}}, \cdots, [T(\vec{b}_n)]_{\mathcal{B}}\big).$$

Then, for all $\vec{x} \in V$, we have

$$[T(\vec{x})]_{\mathcal{B}} = [T]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} \tag{11}$$

STEP 3: Diagonal Matrix Representation. Now, we reach the main result of this part.

Theorem 15. *Suppose two square matrices A and C are similar, that is there exists an invertible matrix P such that $A = PCP^{-1}$, where C is $n \times n$ matrix. Let $\mathcal{B}$ be the basis for $\mathbb{R}^n$ formed from the columns of P, then C is the $\mathcal{B}-$ matrix for the transformation $T : \vec{x} \longrightarrow A\vec{x}$.*

Proof. Let $P = [\vec{b}_1, \cdots, \vec{b}_n]$, so $\mathcal{B} = \{\vec{b}_1, \cdots, \vec{b}_n\}$. Note that

$$\vec{x} = P[\vec{x}]_{\mathcal{B}} \;\longrightarrow\; P^{-1}\vec{x} = [\vec{x}]_{\mathcal{B}}.$$

Also, since we are discussing the linear transformation in $\mathbb{R}^n$, so we have

$$T(\vec{x}) = A\vec{x}.$$

Now we investigate the form of $\mathcal{B}-$matrix. That is

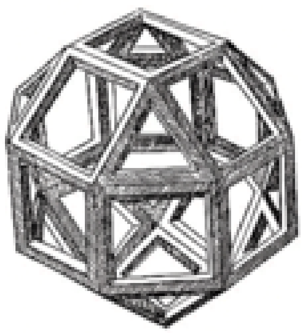
$$\begin{aligned}
[T]_{\mathcal{B}} &= \big([T(\vec{b}_1)]_{\mathcal{B}}, \cdots, [T(\vec{b}_n)]_{\mathcal{B}}\big) \\
&= \big([A\vec{b}_1]_{\mathcal{B}}, \cdots, [A\vec{b}_n]_{\mathcal{B}}\big) \\
&= \big(P^{-1}A\vec{b}_1, \cdots, P^{-1}A\vec{b}_n\big) \\
&= P^{-1}A\big(\vec{b}_1, \cdots, \vec{b}_n\big) \\
&= P^{-1}AP \\
&= C.
\end{aligned} \tag{12}$$

□

Remark. When $C = D$ in the last theorem, it means diagolizing A amounts to finding a diagonal matrix representation of $T : \vec{x} \longrightarrow A\vec{x}$.

5 Complex Eigenvalues

Actually all our preceding introduction can be straightforwardly extended to the complex domain, by changing $\mathbb{R}^n$ to $\mathbb{C}^n$ and changing $\mathbb{R}^{n\times n}$ to $\mathbb{C}^{n\times n}$. When the eigenvalue λ is a complex value, we call it the **complex eigenvalue** and its corresponding vect or.



Rhombicuboctahedron, *by Da Vinci*